

Emergent universe in theories with natural UV cutoffs

Mohsen Khodadi,^{1,*} Kourosh Nozari,^{1,2,†} and Emmanuel N. Saridakis^{3,4,‡}

¹*Department of Physics, Faculty of Basic Sciences,*

University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran

²*Center for Excellence in Astronomy and Astrophysics (CEAAI-RIAAM),*

P. O. Box 55134-441, Maragha, Iran

³*Department of Physics, National Technical University of Athens, Zografou Campus GR 157 73, Athens, Greece*

⁴*CASPER, Physics Department, Baylor University, Waco, TX 76798-7310, USA*

We investigate the realization of the emergent universe scenario in theories with natural UV cutoffs, namely a minimum length and a maximum momentum, quantified by a new deformation parameter in the generalized uncertainty principle. We extract the Einstein static universe solutions and we examine their stability through a phase-space analysis. As we show, the role of the new deformation parameter is crucial in a twofold way. Firstly, it leads to the appearance of new Einstein static universe critical points, that are absent in standard cosmology. Secondly, it plays a central role in providing a mechanism for a graceful exit from a stable Einstein static universe into the expanding thermal history, that is needed for a complete and successful realization of the emergent universe scenario. Finally, we examine the behavior of the scenario under scalar perturbations and we show that the deformation parameter makes it free of perturbative instabilities.

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I. INTRODUCTION

According to the concordance model of cosmology our universe has most probably begun from an initial singularity at a finite past. The introduction of the inflation paradigm as a successful way to solve the horizon, flatness and magnetic monopole problems, did not affect the initial singularity issue, which is still considered as a potential, conceptual disadvantage [1]. Finally, in order to describe the observed late-time universe acceleration a cosmological constant was added, leading eventually to the Λ CDM cosmology, namely the Standard Model of the universe. Nevertheless, in spite of the remarkable successes of this paradigm, its physical content relating to the two accelerating phases at early and late times is still not satisfactory, and furthermore, the initial singularity problem remains open.

There are two ways one could follow in order to bypass the initial singularity problem. The first is to consider the scenario of bouncing cosmology, in which the current universe expansion followed a previous contracting phase, with the scale factor being always non-zero [2, 3]. The second is to consider the scenario of “emergent universe” [4], in which the universe originates from a static state, namely from the “Einstein static universe”, and then it enters the inflationary phase, without passing from any singularity. However, both these alternative cosmological scenarios cannot be obtained in the framework of general relativity. Concerning the Einstein static universe, which is a necessary ingredient of the emergent universe sce-

nario, it can be shown that it is significantly affected by the initial conditions such as perturbations, which dominate at the Ultra-Violet (UV) limit, and hence it is indeed unstable against classical perturbations which eventually lead it to collapse to a singularity [5].

In order to alleviate the above problems one may follow the way to introduce new degree(s) of freedom, beyond the standard model of particle physics or/and general relativity. A first direction is to consider exotic forms of matter that could provide a successful description of the universe behavior in the framework of general relativity (see [6, 7] and references therein). The second direction is to construct a gravitational modification whose extra degrees of freedom could describe the universe at large scales, while still possessing general relativity as a particular limit [8, 9]. Concerning the initial singularity issue, modified gravity, amongst others, can trigger the cosmological bounce [10], or it can cure the emergent universe scenario by making Einstein static universe stable. In particular, the Einstein static universe and thus the emergent universe scenario, can be successfully realized in various gravitational modifications, such as in Einstein-Cartan theory [11], in $f(R)$ gravity [12], in $f(T)$ gravity [13], in loop quantum cosmology [14], in massive gravity [15], in Horava-Lifshitz gravity [16], in braneworld models [17], etc, although the successful exit from the Einstein static universe towards the subsequent expanding thermal history is not always achieved.

One interesting gravitational modification in the UV regime arises through the use of the “generalized uncertainty principle” [18], which seen as a quantum gravity approach might be related with other quantum gravity models such as Double Special Relativity [19] and string theory [20]. Although one can have more than one generalizations of the uncertainty principle, the most interesting one is when one modifies the standard Heisenberg

*Electronic address: m.khodadi@stu.umz.ac.ir

†Electronic address: knozari@umz.ac.ir

‡Electronic address: Emmanuel.Saridakis@baylor.edu

algebra by a linear and a quadratic term in Planck length and momentum respectively, which leads to the existence of two natural UV cutoffs, namely minimum length and maximum momentum [21]. Hence, when applied to a cosmological framework, these natural cutoffs give rise to extra terms in the Friedmann equations, which can have interesting implications.

In the present work we are interested in investigating the Einstein static universe and the emergent universe scenario in the framework of theories with natural UV cutoffs. In particular, we show how the induced extra terms in the cosmological equations lead to the realization and stability of the Einstein static universe, as well as offering the mechanism to a phase transition to the inflationary era and the subsequent thermal history of the universe.

The plan of the manuscript is the following: In section II, we briefly review generalized uncertainty principle with two UV cutoffs, and we apply it in a cosmological framework. In section III, we examine the stability of the Einstein static universe by performing a dynamical system analysis, studying its exit towards the inflationary era. In section IV we extract the conditions under which the scenario is stable against scalar perturbations. Finally, in section V we provide the conclusions.

II. THEORIES WITH NATURAL UV CUTOFFS AND THEIR COSMOLOGY

In this section we briefly present theories with natural UV cutoffs, and then we apply them in a cosmological framework. As we mentioned in the Introduction, in general this kind of theories arise from the consideration of generalizations of the uncertainty principle [18]. Although one may have more than one such generalizations, in the present work we focus on the generalization with two natural UV cutoffs, namely a minimum length and a maximum momentum [21]. In order to achieve this, one starts by modifying the standard Heisenberg algebra at high energy scales, by a linear and a quadratic term in Planck length and momentum respectively, as

$$[x_i, p_j] = i\hbar \left[\delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j) \right], \quad (1)$$

with $i, j = 1, 2, 3$, where via the Jacobi identity [22] it is guaranteed that $[x_i, x_j] = 0 = [p_i, p_j]$. The parameter α quantifies the quantum gravity deformation parameter, and can alternatively be written as $\alpha = \frac{\tilde{\alpha}}{cM_{pl}} = \frac{\ell_{pl}c^2}{\hbar}\tilde{\alpha}$, with M_{pl} and ℓ_{pl} the Planck mass and Planck length respectively, c the speed of light, and \hbar the induced Planck constant. The dimensionless parameter $\tilde{\alpha}$ according to experiments is bound to be smaller than 10^{11} [23, 24].

Let us now apply the above generalized uncertainty principle in a cosmological framework. Since the quantum gravity deformation parameter α is expected to have effects only at high energy scales, we will focus on the

early-time phases of the cosmological evolution, which indeed will correspond to the realization of the emergent universe. We start by considering the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) geometry, with metric

$$ds^2 = -N^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - k r^2} + r^2 d\Omega^2 \right), \quad (2)$$

where $a(t)$ is the scale factor and N the lapse function, and with $k = 0, +1, -1$ corresponding to flat, close and open spatial geometry respectively.

One can extract the field equations in the above metric, i.e. the Friedmann equations, via the Hamiltonian constraint $\mathcal{H}_E = 0$, namely [25]:

$$\mathcal{H}_E = \frac{\kappa}{4} \frac{N p_a^2}{a} + \frac{N a k}{\kappa} - N a^3 \rho + \lambda \mathcal{P}, \quad (3)$$

with $\kappa \equiv 1/3M_{pl}^2 = 8\pi G/3$ the gravitational constant, and where λ and \mathcal{P} are the Lagrange multiplier and the momentum conjugate to the lapse function N , respectively. In the above expression ρ is the energy density of the universe content, corresponding to a perfect fluid with equation-of-state parameter w .

In general, for two typical variables A and B , the Poisson brackets are defined as $\{A, B\} = \left(\frac{\partial A}{\partial x_i} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_j} \right) \{x_i, p_j\}$, where the canonical variables x_i and p_j in the cosmological context are replaced by a and p_a , respectively. Although using the standard uncertainty principle they satisfy the usual relation $\{a, p_a\} = 1$, considering the deformed Poisson algebra that arises from the generalized uncertainty principle (1), up to first order in α , the Poisson bracket between a and p_a becomes [21]

$$\{a, p_a\} = 1 - 2\alpha p_a. \quad (4)$$

Hence, using the Poisson algebra we obtain the following modified equations of motion

$$\dot{a} = \{a, \mathcal{H}_E\} = \frac{\partial \mathcal{H}_E}{\partial p_a} (1 - 2\alpha p_a), \quad (5)$$

$$\dot{p}_a = \{p_a, \mathcal{H}_E\} = -\frac{\partial \mathcal{H}_E}{\partial a} (1 - 2\alpha p_a). \quad (6)$$

Inserting \mathcal{H}_E from (3) in the above equations, using its constraint value $\mathcal{H}_E = 0$, and combining them, we finally extract the first Friedmann equation, namely

$$\left(\frac{\dot{a}}{a} \right)^2 = \kappa \rho - \frac{\kappa c^2}{a^2} - 2\sqrt{2\kappa} \alpha a^2 \rho^{3/2} \left(1 - \frac{\kappa c^2}{\kappa a^2 \rho} \right)^{3/2}. \quad (7)$$

Additionally, taking the time-derivative of this equation, and using also the usual energy conservation relation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1 + w) = 0, \quad (8)$$

we arrive at the second Friedmann equation, namely

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{2}(1+3w)\rho - 7\sqrt{\frac{\kappa}{2}}\alpha a^2 \rho^{3/2} \left(1 - \frac{\kappa c^2}{\kappa a^2 \rho}\right)^{3/2} + 3\alpha a^2 \rho^{1/2} \left[\sqrt{\frac{\kappa}{2}}(1+3w)\rho - \sqrt{\frac{2}{\kappa}} \frac{\kappa c^2}{a^2} \right] \left(1 - \frac{\kappa c^2}{\kappa a^2 \rho}\right)^{1/2}. \quad (9)$$

As we observe, the two Friedmann equations (7) and (9) include terms with the quantum gravity deformation parameter α , i.e they have been modified by the generalized uncertainty principle. As expected, in the limit $\alpha \rightarrow 0$ they give rise to the standard Friedmann equations.

III. EINSTEIN STATIC UNIVERSE AND ITS DYNAMICAL STABILITY

In this section we first show that in the cosmological application of the generalized uncertainty principle the Einstein static universe can be realized. Then, we express the cosmological equations as a first-order dynamical system and we perform a detailed phase-space analysis in order to examine its stability, i.e to see whether the universe can remain in such a phase for very large time intervals.

Let us first extract the Einstein static universe solutions. Inserting the conditions of the Einstein static universe, i.e. a constant scale factor $a = a_s$, with $\ddot{a}|_{a=a_s} = \dot{a}|_{a=a_s} = 0$, at an energy density $\rho = \rho_s$, in the two Friedmann equations (7) and (9), and focusing for mathematical convenience (although this is not necessary) on the regime $\rho \gg \frac{|k|c^2}{\kappa}$ (which is a very robust approximation since in SI units it becomes $\rho \gg |k| \times 10^{27} \text{ kg}^{-3}$, while we know that the energy density corresponding to (pre)-inflation scale is $\sim 10^{93} \text{ kg m}^{-3}$) we find

$$\kappa \rho_s - \frac{\kappa c^2}{a_s^2} + \frac{6\alpha \kappa c^2}{\sqrt{2\kappa}} \rho_s^{1/2} - 2\sqrt{2\kappa} \alpha a_s^2 \rho_s^{3/2} = 0, \quad (10)$$

$$\frac{3k^2 c^4 \alpha}{\sqrt{2\kappa^3 \rho_s}} \frac{1}{a_s^4} + \left[\frac{3\kappa c^2}{2\sqrt{2\kappa}} (2-w)\alpha - \frac{\kappa}{2} (1+3w)\rho_s \right] \frac{1}{a_s^2} - 2\sqrt{2\kappa} \alpha \rho_s^{3/2} = 0. \quad (11)$$

The solution of this system of algebraic equations will give the critical points of the cosmological scenario at hand, namely the pair of values for $\{a_s, \rho_s\}$ that correspond to Einstein static universe solutions. Hence, performing linear perturbations around them, and examining the eigenvalues of the involved perturbation matrix, will reveal whether these solutions are stable or unstable [26]. In the following two subsections we examine the flat and non-flat cases separately.

A. Flat universe ($k = 0$)

In the case of a flat geometry, and for a general $w \neq -1$, the system (10),(11) accepts only the trivial solution

$a_s \rightarrow \infty, \rho_s \rightarrow 0$, independently of the values of α . However, in the special case where $w = -1$, i.e where the universe is filled with a cosmological constant, we obtain an Einstein static universe solution for every ρ_s , with the corresponding scale factor being

$$\frac{1}{a_s^2} = \frac{\sqrt{\kappa}}{2\alpha\sqrt{2}\sqrt{\rho_s}}. \quad (12)$$

Note that the role of a non-zero quantum gravity deformation parameter α is crucial in making the above solution non-trivial, since for $\alpha = 0$ one obtains only the aforementioned trivial solution.

Let us now study the dynamical stability of this solution, following the usual procedure of dynamical system analysis [26]. We introduce two variables, namely $x_1 = a$ and $x_2 = \dot{a}$, and hence the linear perturbations of the Friedmann equation (9), around the critical point (12) and with $w = -1$, leads to

$$\dot{x}_1 = x_2 \equiv O_1(x_1, x_2), \quad (13)$$

$$\dot{x}_2 = \kappa \rho_s x_1 - 2\sqrt{2\kappa} \alpha \rho_s^{3/2} x_1^3 \equiv O_2(x_1, x_2). \quad (14)$$

Hence, the eigenvalues square λ^2 of the Jacobian matrix

$$J \left(O_1(x_1, x_2), O_2(x_1, x_2) \right) = \begin{pmatrix} \frac{\partial O_1}{\partial x_1} & \frac{\partial O_1}{\partial x_2} \\ \frac{\partial O_2}{\partial x_1} & \frac{\partial O_2}{\partial x_2} \end{pmatrix}, \quad (15)$$

calculated at the critical point (12), is just

$$\lambda^2 = \kappa \rho_s - 24\alpha^2 \rho_s^2. \quad (16)$$

As usual, if the above eigenvalues square is negative then the corresponding critical point is stable, that is any small perturbation around this solution will decrease and the system will return to it. Interestingly enough we observe that while for $\alpha \rightarrow 0$ we obtain the usual result that $\lambda^2 > 0$, and therefore that the Einstein static universe solution in cosmology with usual uncertainty principle is unstable, for a suitably large α , namely for $\alpha^2 > \kappa/(24\rho_s)$, we acquire $\lambda^2 < 0$ and thus the Einstein static universe becomes stable. Hence, we can clearly see the crucial effect of the generalized uncertainty principle on the stability of the Einstein static universe.

In order to see the above effect more transparently, in the upper graph of Fig. 1 we present the phase-space behavior for the spatially flat cosmology with equation-of-state parameter $w = -1$, where the stable Einstein static universe critical point is clear (the physical critical point is the one with $x_1 > 0$). Moreover, in order to verify the stability of the Einstein static universe solution in an alternative way, in the lower graph of Fig. 1 we depict the evolution of the scale factor after a small perturbation around this solution. As can be seen, the universe exhibits small oscillations around the Einstein static universe, without deviating from it, as expected.

In summary, as we can see from the simple case of flat geometry, the effect of the quantum gravity deformation parameter that arise from the generalized uncertainty

principle is twofold: Firstly, it leads to a non-trivial Einstein static universe solution that is absent in standard cosmological models, and secondly it leads to its stabilization. This is one of the main results of the present work, and will become more transparent in the more interesting solutions in the case of non-flat geometry.

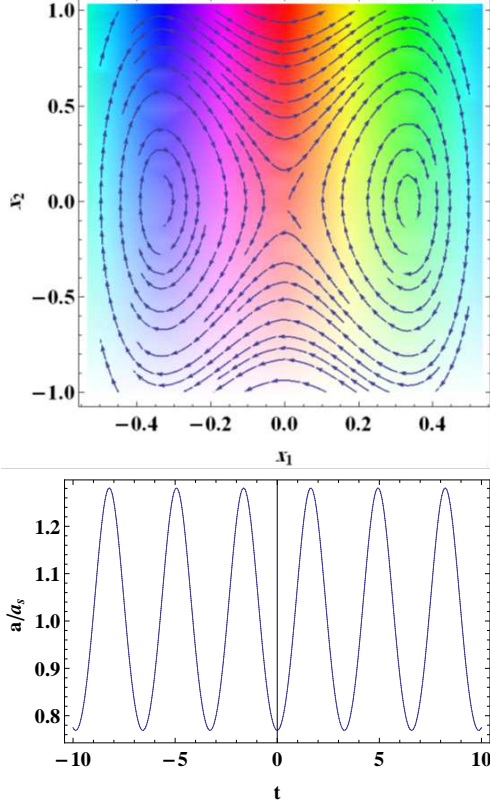


FIG. 1: The phase diagram in (a, \dot{a}) or (x_1, x_2) space (upper graph) and the evolution of the scale factor (lower graph), for the spatially flat cosmology, with equation-of-state parameter $w = -1$, for the choice $\alpha = 1$, in units where $c = \kappa = \ell_p = 1$. The value of ρ_s has been chosen as $\rho_s = 10^2$, in order to be consistent with the condition $\rho \gg \frac{|k|c^2}{\kappa}$.

B. Non-flat universe ($k \neq 0$)

Let us now investigate the non-flat universe. In this case, the system (10),(11) accepts four solutions, i.e four Critical Points (CP), namely

$$\text{CP 1: } \left(\frac{1}{a_s^2} \right)_1 = 0, \quad (17)$$

$$\text{CP 2: } \left(\frac{1}{a_s^2} \right)_2 = -\frac{3\alpha^2 k c^2 (2-w)}{\kappa^2 (1+3w)} \times \left\{ -1 + \sqrt{1 + \frac{32}{3} \left[\frac{1+3w}{(2-w)^2} \right]} \right\}, \quad (18)$$

$$\text{CP 3: } \left(\frac{1}{a_s^2} \right)_3 = -\frac{3\alpha^2 k c^2 (2-w)}{\kappa^2 (1+3w)} \times \left\{ -1 - \sqrt{1 + \frac{32}{3} \left[\frac{1+3w}{(2-w)^2} \right]} \right\}, \quad (19)$$

$$\text{CP 4: } \left(\frac{1}{a_s^2} \right)_4 = -2\sqrt{\frac{q_1}{3}} \sinh\left(\frac{q_2}{3}\right), \quad (20)$$

with

$$\begin{aligned} q_1 &= -\frac{12k^2 c^4 \alpha^4}{\kappa^4} \left[\frac{w^2 + 20w + 12}{(1+3w)^2} \right], \\ q_2 &= \sinh^{-1} \left(\frac{3q_3}{q_4 q_1} \right), \\ q_3 &= \frac{36k^3 c^6 \alpha^6}{\kappa^4} \frac{(2-w)}{(1+3w)^2} \left[1 - \frac{4}{9} \frac{(2-w)^2}{1+3w} \right], \\ q_4 &= \frac{4k c^2 \alpha^2}{\kappa^2 (1+3w)} \sqrt{-w^2 - 20w - 12}. \end{aligned} \quad (21)$$

The critical point 1 is the trivial one with $a_s \rightarrow \infty$. The critical point 2 is physical for $k = +1$, $w > 2$ or $k = -1$, $-\frac{1}{3} < w < 2$, with the second case being the realistic one. The critical point 3 is physical for $k = +1$, $-\frac{1}{3} < w < 2$ or $k = -1$, $w > 2$, with the first case being the realistic one. Finally, point 4 is physical for $-10 - 2\sqrt{22} < w < -10 + 2\sqrt{22}$, which includes the cosmological constant value $w = -1$. We stress once again the crucial role of the deformation parameter α that arises from the generalized uncertainty principle, in the existence of the three non-trivial critical points, since in the limit $\alpha \rightarrow 0$ all points coincide with the first, trivial one.

For each one of the above solutions for a_s , the corresponding ρ_s is given by

$$\rho_s^{1/2} = -\frac{1}{3A} \left(\kappa + \Delta_2 + \frac{\Delta_0}{\Delta_2} \right), \quad (22)$$

where

$$\begin{aligned} \Delta_2 &= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \\ \Delta_0 &\equiv \kappa^2 - 3AB, \\ \Delta_1 &\equiv 2\kappa^3 - 9\kappa AB + 27A^2 C, \end{aligned} \quad (23)$$

with

$$A \equiv -2\sqrt{2\kappa}\alpha a_s^2, \quad B \equiv \frac{6\alpha k c^2}{\sqrt{2\kappa}}, \quad C \equiv -\frac{k c^2}{a_s^2}. \quad (24)$$

We now proceed to the investigation of the dynamical stability of the above solutions, following the usual procedure of dynamical system analysis [26], similarly to the flat case. We introduce the two variables $x_1 = a$ and $x_2 = \dot{a}$, and therefore the linear perturbations of the Friedmann equation (9) around the critical points (17)-

(20) leads to

$$\dot{x}_1 = x_2 \equiv O_1(x_1, x_2), \quad (25)$$

$$\ddot{x}_2 = \left[\frac{3kc^2}{2\sqrt{2}\kappa}(2-w)\alpha - \frac{\kappa}{2}(1+3w)\rho_s \right] x_1 - 2\sqrt{2}\kappa\alpha\rho_s^{3/2}x_1^3 + \frac{3k^2c^4\alpha}{\sqrt{2}\kappa^3\rho_s} \frac{1}{x_1} \equiv O_2(x_1, x_2). \quad (26)$$

The eigenvalues square of the Jacobian matrix (15), for the above $O_1(x_1, x_2), O_2(x_1, x_2)$, calculated at the non-trivial critical points 2, 3 and 4, read as follows.

For the points 2 and 3 we have

$$\lambda^2 = -\frac{9\alpha^2k^2c^4}{16\kappa^2} \left[\frac{(2-w)^2}{1+3w} \right] [2f(w)_\pm^2 + f(w)_\pm] + \frac{2\kappa^4}{9\alpha^4kc^2} \left(\frac{1+3w}{2-w} \right)^3 f(w)_\pm^{-3}, \quad (27)$$

with $f(w)_\pm \equiv -1 \pm \sqrt{1 + \frac{32}{3} \left[\frac{1+3w}{(2-w)^2} \right]}$, where the plus sign corresponds to critical point 2 and the minus sign to critical point 3. As we can see, for the range where they are physical, namely for $k = +1$, $w > 2$ or $k = -1$, $-\frac{1}{3} < w < 2$ for critical point 2, and for $k = +1$, $-\frac{1}{3} < w < 2$ or $k = -1$, $w > 2$ for critical point 3, we always get $\lambda^2 < 0$, and thus both points are always stable.

Let us now examine the stability of the critical point 4 given in (20). The corresponding eigenvalues square of the Jacobian matrix is found to be

$$\lambda^2 = -\frac{9\alpha^2}{2\sqrt{2}\kappa^2q_1 \sinh^2\left(\frac{q_2}{3}\right)} + \frac{3kc^2(w-2)}{16} \sqrt{\frac{q_1}{3}} \sinh\left(\frac{q_2}{3}\right) - \frac{\kappa^2q_1}{12\alpha^2}(1+3w) \sinh^2\left(\frac{q_2}{3}\right). \quad (28)$$

Hence, we deduce that the sign of λ^2 depends on both α and w in a complicated way that does not allow for an analytical treatment, and thus in order to examine its behavior we will resort to numerical elaboration. This point exhibits a very interesting behavior concerning the realization of the emergent universe scenario. In particular, for suitable values of α it is stable as long as w is smaller than a specific value, and it becomes unstable for larger w values. Therefore, we can have a stable Einstein static universe for very long time intervals (infinite in the past if w approaches -1 in the past), and as time passes and the universe equation-of-state parameter increases this critical point becomes unstable, offering a natural graceful exit from the Einstein static universe and an entering into the usual expanding thermal history, i.e. a successful realization of the emergent universe scenario. This behavior is also achieved in complicated models of the emergent universe in various modified gravities [27], however in the present scenario it is obtained solely from the quantum gravity deformation parameter α .

In order to see the above behavior more transparently, in Fig. 2 we depict λ^2 versus w for various values of

α , for the case of closed and open geometry. As can be observed from the two graphs, the role of α is crucial in inducing the successful exit from the Einstein static universe, since in the limit $\alpha \rightarrow 0$ the universe cannot exit from it and thus the subsequent thermal history is not possible. Hence, we can clearly deduce the central role of the quantum gravity deformation parameter that arise from the generalized uncertainty principle: Firstly, it leads to a non-trivial Einstein static universe solution that is absent in standard cosmological models, and secondly it provides a mechanism for a successful exit from a stable Einstein static universe into the expanding thermal history, i.e. for a complete realization of the emergent universe scenario. This is one of the main results of the present work.

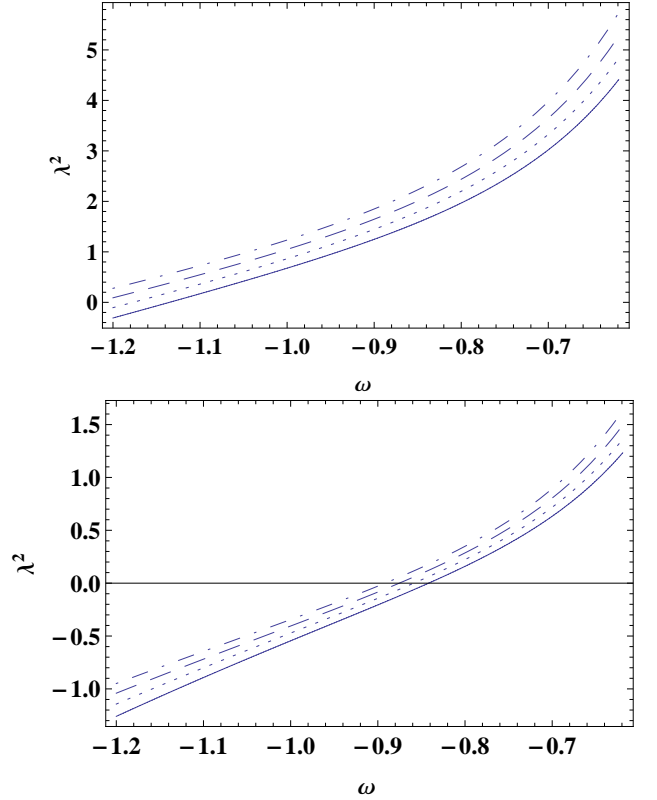


FIG. 2: The eigenvalues square λ^2 versus the equation-of-state parameter w for the critical point 4 given in (20), for closed (upper graph) and open (lower graph) geometry, for various values of α , namely $\alpha = 0.88$ (solid curve) $\alpha = 0.92$ (dotted curve) $\alpha = 0.96$ (dashed curve) $\alpha = 1$ (dashed-dot curve), in units where $c = \kappa = \ell_p = 1$.

IV. SCALAR PERTURBATIONS AND STABILITY CONDITIONS

In this section we are interested in performing an analysis of the scalar perturbations in the scenario of cosmology under the generalized uncertainty principle. Specifically, we want to extract conditions under which the

cosmological evolution is free of perturbative instabilities and thus physical. We mention here that the notion of stability in this section is used differently than that in the dynamical system analysis of the previous section. In particular, while in dynamical systems an unstable solution is a physical solution that just cannot attract the universe, in the present perturbation analysis a solution with perturbative instabilities implies that it is ill-behaved and therefore not physical.

In order to perform the analysis of scalar perturbations we perturb linearly the Friedmann equations (7) and (9) in the regime $\rho \gg \frac{|k|c^2}{\kappa}$, around the obtained Einstein static universe solutions (12) and (17)-(20). In particular, the perturbations in the scale factor and matter density read as:

$$\begin{aligned} a(t) &\rightarrow a_s(1 + \delta a(t)), \\ \rho(t) &\rightarrow \rho_s(1 + \delta \rho(t)). \end{aligned} \quad (29)$$

Inserting into the first Friedmann equation (7), using

$$\begin{aligned} (1 + \delta a(t))^n &\simeq 1 + n\delta a(t), \\ (1 + \delta \rho(t))^n &\simeq 1 + n\delta \rho(t), \end{aligned} \quad (30)$$

and neglecting terms with two differentials, we obtain

$$\begin{aligned} \kappa\rho_s(1 + \delta\rho) - kc^2a_s^{-2} - 2\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2} + 2kc^2a_s^{-2}\delta a \\ - \frac{3\alpha kc^2}{\sqrt{2\kappa}}\rho_s^{1/2}(2 + \delta\rho) - \sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2}(3\delta\rho + 4\delta a) = 0. \end{aligned} \quad (31)$$

Similarly, using (29) to perturb the second Friedmann equation (9), and neglecting terms with two differentials, we obtain

$$\begin{aligned} \delta\ddot{a} = &\left(4\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2} + \frac{126\alpha k^2c^4}{\sqrt{(2\kappa)^3}}a_s^{-2}\rho_s^{-1/2}\right)\delta a \\ &- \left[3\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2} - \frac{3\alpha kc^2}{\sqrt{2\kappa}}(2 - w)\rho_s^{1/2}\right. \\ &\left.+ \frac{3\alpha k^2c^4}{\sqrt{(2\kappa)^3}}a_s^{-2}\rho_s^{-1/2} + \frac{\kappa}{2}(3w + 1)\rho_s\right]\delta\rho. \end{aligned} \quad (32)$$

In the following two subsections we examine the flat and non-flat cases separately.

A. Flat universe ($k = 0$)

In the case of a flat universe, (31) leads to

$$\left(\frac{\delta\rho}{\delta a}\right) = \left(\frac{4\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2}}{\kappa\rho_s - 3\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2}}\right). \quad (33)$$

Thus, inserting (33) into (32) and neglecting terms higher than $\mathcal{O}(\alpha^2)$, we acquire

$$\delta\ddot{a} + \gamma_f\delta a = 0, \quad (34)$$

with

$$\gamma_f = \frac{6\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2}(w + 1)}{\kappa - 3\sqrt{2\kappa\alpha}a_s^2\rho_s^{1/2}}. \quad (35)$$

However, as we found in subsection III A, the non-trivial Einstein static solution (12) exists only for $w = -1$, which leads to the limiting value $\gamma_f = 0$. Hence, we deduce that the scenario at hand might not be stable against ghost instabilities.

B. Non-flat universe ($k \neq 0$)

In the case of a non-flat universe, (31) leads to

$$\left(\frac{\delta\rho}{\delta a}\right) = \left(\frac{4\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2} - 2kc^2a_s^{-2}}{\kappa\rho_s - \frac{3\alpha kc^2}{\sqrt{2\kappa}}\rho_s^{1/2} - 3\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2}}\right). \quad (36)$$

As expected, in the limit $\alpha \rightarrow 0$ we re-obtain the standard result, namely $\left(\frac{\delta\rho}{\delta a}\right) = -\frac{2kc^2}{a_s^2\kappa\rho_s}$. Replacing (36) into (32) and neglecting terms higher than $\mathcal{O}(\alpha^2)$, we acquire

$$\delta\ddot{a} + \gamma_{nf}\delta a = 0, \quad (37)$$

with

$$\gamma_{nf} = \left[\frac{\kappa\rho_s(f_1 + f_2) + f_3 + f_4 + f_5 + f_6 + f_7}{f_8}\right], \quad (38)$$

and

$$\begin{aligned} f_1 &= 4\sqrt{2\kappa\alpha}a_s^2\rho_s^{3/2}, \\ f_2 &= \frac{12\alpha k^2c^4}{\sqrt{8\kappa^3}}a_s^{-2}\rho_s^{-1/2}, \\ f_3 &= 2\sqrt{2\kappa^3\alpha}(3w + 1)a_s^2\rho_s^{5/2}, \\ f_4 &= -6\sqrt{2\kappa\alpha}kc^2\rho_s^{3/2}, \\ f_5 &= -\frac{6\alpha k^3c^6}{\sqrt{8\kappa^3}\rho_s a_s^4}, \\ f_6 &= \frac{6\alpha k^2c^4\rho_s^{1/2}(2 - w)}{\sqrt{2\kappa}a_s^2}, \\ f_7 &= -kc^2\kappa(3w + 1)\rho_s a_s^{-2}, \\ f_8 &= -\frac{3}{4}f_1 + \frac{3\alpha kc^2}{\sqrt{2\kappa}}\rho_s^{1/2} + \kappa\rho_s, \end{aligned} \quad (39)$$

and where a_s and ρ_s have to be replaced from (17)-(20) for the four Einstein static universes respectively. Equation (37) is the perturbation equation of FRW cosmology in the case of generalized uncertainty principle. As expected, in the limit $\alpha \rightarrow 0$ it reduces to the standard result, namely

$$\delta\ddot{a} - \frac{kc^2}{a_s^2}(3w + 1)\delta a_s = 0. \quad (40)$$

Let us now examine whether the scenario at hand is free of perturbative instabilities, namely whether we can obtain $\gamma_{nf} > 0$. The form of γ_{nf} given in (38), calculated at the non-trivial critical points (18)-(20), is too complicated to accept any analytical treatment, and thus we will examine the value of γ_{nf} numerically. In Fig. 3 we

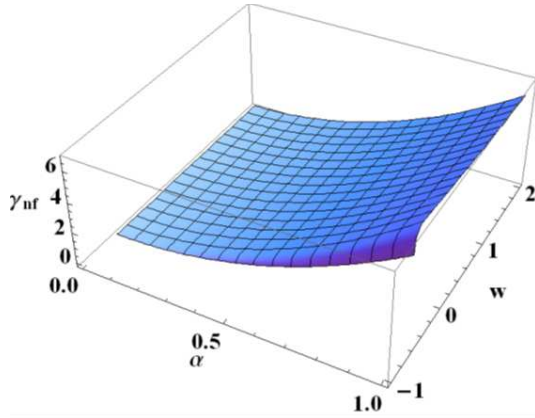


FIG. 3: The coefficient of the perturbation equation γ_{nf} versus the quantum gravity deformation parameter α and the equation-of-state parameter w , for the case of critical point 2 of (18), in the case of open geometry, in units where $c = \kappa = \ell_p = 1$.

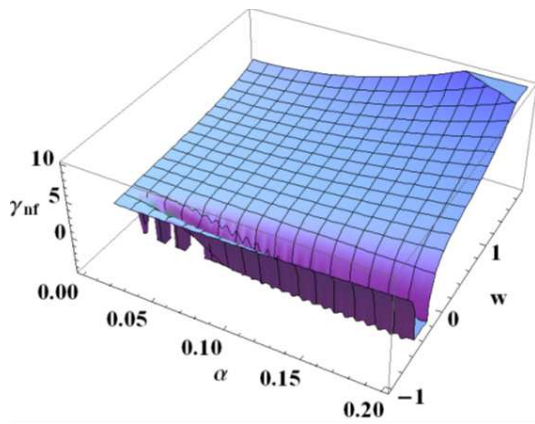


FIG. 4: The coefficient of the perturbation equation γ_{nf} versus the quantum gravity deformation parameter α and the equation-of-state parameter w , for the case of critical point 3 of (19), in the case of closed geometry, in units where $c = \kappa = \ell_p = 1$.

present γ_{nf} versus α and w , for the case of critical point 2 of (18), in the case of open geometry (since for the open geometry this point exists for the more physically interesting w -interval, namely $-\frac{1}{3} < w < 2$). As we can see, for the regions of its existence, we obtain $\gamma_{nf} > 0$ if α acquires positive values. Hence, the scenario at hand can be free of perturbative instabilities. Similarly, in Fig. 4 we present γ_{nf} versus α and w , for the critical point 3 of (19), in the case of closed geometry (where $-\frac{1}{3} < w < 2$).

As we observe, this point can be free of perturbative instabilities for suitable choices of α and w . Finally, in Fig. 5 we present γ_{nf} versus α and w , for the critical point 4 of (20), for a part of the range of its existence, namely for $-10 - 2\sqrt{22} < w < -10 + 2\sqrt{22}$, in the case of closed and open geometry. Similarly to the previous critical points, we can see that for suitable values of α and w this point, which is the most interesting one concerning the successful realization of the emergent universe scenario, is free of perturbative instabilities.

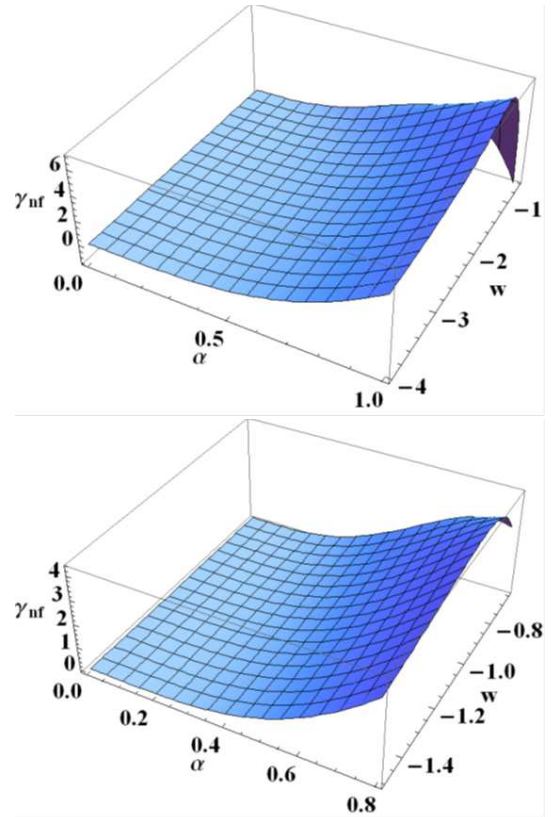


FIG. 5: The coefficient of the perturbation equation γ_{nf} versus the quantum gravity deformation parameter α and the equation-of-state parameter w , for the case of critical point 4 of (20), in the case of closed (upper graph) and open (lower graph) geometry, in units where $c = \kappa = \ell_p = 1$.

We close this section by mentioning that in the limit $\alpha \rightarrow 0$ we acquire $\gamma_{nf} \rightarrow 0$, i.e the (trivial in this case) Einstein static universe in standard cosmology suffers from ghost instabilities, as expected. Once again we observe that the role of the non-zero quantum gravity deformation parameter α is crucial, since apart from inducing new, non-trivial Einstein static universe solutions, it additionally makes them free of ghost instabilities.

V. DISCUSSIONS AND CONCLUSIONS

In the present work we investigated the realization of the emergent universe scenario in the framework of theo-

ries with natural UV cutoffs. In particular, we considered the generalized uncertainty principle, which includes a deformation parameter α that arises from quantum gravity modifications corresponding to two natural cutoffs, namely a minimum length and a maximum momentum. Applying it in a cosmological framework we obtained the modified Friedmann equations and we studied them in detail to see whether we can acquire Einstein static universe solutions, which are the basic concept in the realization of the emergent universe scenario.

As a first step we extracted the Einstein static universe solutions, analyzing for convenience the flat and non-flat cases separately. Additionally, performing a dynamical analysis in the phase space we examined the dynamical stability of these solutions. As we showed, the role of the new deformation parameter α is crucial in a twofold way. Firstly, it leads to the appearance of new Einstein static universe critical points, that are absent in standard cosmology, and this is true also in the flat case where standard cosmology does not accept an Einstein static universe. Secondly, the deformation parameter α plays a central role in providing a mechanism for a graceful exit from a stable Einstein static universe into the expanding thermal history, i.e. for a complete and successful realization of the emergent universe scenario. This double role of α , i.e. of the quantum gravity modifications arising from the natural UV cutoffs, is one of the main results of the present work.

As a second step we investigated the behavior of the

model under scalar perturbations, aiming to examine whether the scenario at hand is free of perturbative instabilities such as ghost or Laplacian instabilities. Deriving the perturbation equations for the flat and non-flat cases, we showed that the obtained background Einstein static universe solutions are free of perturbative instabilities. Once again we observed that the role of the non-zero quantum gravity deformation parameter α is crucial, since apart from inducing new, non-trivial Einstein static universe solutions, it makes them free of ghost instabilities, too.

In summary, we conclude that the emergent universe scenario can be successfully realized in the framework of cosmology with generalized uncertainty principle arising from the presence of natural UV cutoffs.

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- [1] A. H. Guth, *Phys. Rev. D* **23**, 347 (1981); K. Sato, *Mon. Not. Roy. Astron. Soc.* **195**, 467 (1981). A. D. Linde, *Phys. Lett.* **108B**, 389 (1982).
 - [2] V. F. Mukhanov and R. H. Brandenberger, *Phys. Rev. Lett.* **68**, 1969 (1992).
 - [3] M. Novello and S. E. P. Bergliaffa, *Phys. Rept.* **463**, 127 (2008).
 - [4] G. F. R. Ellis and R. Maartens, *Class. Quant. Grav.* **21**, 223 (2004); G. F. R. Ellis, J. Murugan and C. G. Tsagas, *Class. Quant. Grav.* **21**, no. 1, 233 (2004).
 - [5] A. S. Eddington, *Mon. Not. Roy. Astron. Soc.* **90**, 668 (1930); G. W. Gibbons, *Nucl. Phys. B* **310**, 636 (1988).
 - [6] E. J. Copeland, M. Sami and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006).
 - [7] Y. F. Cai, E. N. Saridakis, M. R. Setare and J. Q. Xia, *Phys. Rept.* **493**, 1 (2010).
 - [8] S. Nojiri and S. D. Odintsov, *eConf C0602061*, 06 (2006), *Int. J. Geom. Meth. Mod. Phys.* **4**, 115 (2007).
 - [9] S. Capozziello and M. De Laurentis, *Phys. Rept.* **509**, 167 (2011).
 - [10] R. H. Brandenberger, V. F. Mukhanov and A. Sornborger, *Phys. Rev. D* **48**, 1629 (1993); J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, *Phys. Rev. D* **65**, 086007 (2002); M. Bojowald, *Phys. Rev. Lett.* **86**, 5227 (2001); J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, *Phys. Rev. D* **64**, 123522 (2001); Y. Shtanov and V. Sahni, *Phys. Lett. B* **557**, 1 (2003); J. Martin and P. Peter, *Phys. Rev. D* **68**, 103517 (2003); P. Creminelli and L. Senatore, *JCAP* **0711**, 010 (2007); E. N. Saridakis, *Nucl. Phys. B* **808**, 224 (2009); J. L. Lehnert, *Phys. Rept.* **465**, 223 (2008); R. Brandenberger, *Phys. Rev. D* **80**, 043516 (2009); Y. -F. Cai and E. N. Saridakis, *Class. Quant. Grav.* **28**, 035010 (2011); Y. F. Cai, D. A. Easson and R. Brandenberger, *JCAP* **1208**, 020 (2012); T. Biswas, A. S. Koshelev, A. Mazumdar and S. Y. Vernov, *JCAP* **1208**, 024 (2012); Y. F. Cai, C. Gao and E. N. Saridakis, *JCAP* **1210** (2012) 048; T. Qiu, X. Gao and E. N. Saridakis, *Phys. Rev. D* **88**, no. 4, 043525 (2013); Y. F. Cai, J. Quintin, E. N. Saridakis and E. Wilson-Ewing, *JCAP* **1407**, 033 (2014).
 - [11] C. G. Boehmer, *Class. Quant. Grav.* **21**, 1119 (2004); K. Atazadeh, *JCAP* **1406**, 020 (2014).
 - [12] J. D. Barrow and A. C. Ottewill, *J. Phys. A* **16**, 2757 (1983); C. G. Boehmer, L. Hollenstein and F. S. N. Lobo, *Phys. Rev. D* **76**, 084005 (2007); R. Goswami, N. Goheer and P. K. S. Dunsby, *Phys. Rev. D* **78**, 044011 (2008).
 - [13] P. Wu and H. Yu, *Phys. Lett. B* **703**, 223 (2011); J. T. Li, C. C. Lee and C. Q. Geng, *Eur. Phys. J. C* **73**, no. 2, 2315 (2013).
 - [14] D. J. Mulryne, R. Tavakol, J. E. Lidsey and G. F. R. Ellis, *Phys. Rev. D* **71**, 123512 (2005); L. Parisi, M. Bruni, R. Maartens and K. Vandersloot, *Class. Quant. Grav.* **24**, 6243 (2007).
 - [15] L. Parisi, N. Radicella and G. Vilasi, *Phys. Rev. D* **86**, 024035 (2012); K. Zhang, P. Wu and H. Yu, *Phys. Rev. D* **87**, no. 6, 063513 (2013).

- [16] M. Khodadi, Y. Heydarzade, F. Darabi and E. N. Saridakis, Phys. Rev. D **93**, no. 12, 124019 (2016); Y. Heydarzade, M. Khodadi and F. Darabi, arXiv:1502.04445 [gr-qc].
- [17] L. A. Gergely and R. Maartens, Class. Quant. Grav. **19**, 213 (2002); K. Zhang, P. Wu and H. W. Yu, Phys. Lett. B **690**, 229 (2010); K. Atazadeh, Y. Heydarzade and F. Darabi, Phys. Lett. B **732**, 223 (2014); Y. Heydarzade and F. Darabi, JCAP **1504**, no. 04, 028 (2015).
- [18] S. Hossenfelder, Living Rev. Rel. **16**, 2 (2013).
- [19] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002); G. Amelino-Camelia, Phys. Lett. B **510**, 255 (2001); J. Magueijo and L. Smolin, Class. Quant. Grav. **21**, 1725 (2004); J. Magueijo and L. Smolin, Phys. Rev. D **67**, 044017 (2003).
- [20] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B **216**, 41 (1989).
- [21] A. F. Ali, S. Das and E. C. Vagenas, Phys. Lett. B **678**, 497 (2009); S. Das, E. C. Vagenas and A. F. Ali, Phys. Lett. B **690**, 407 (2010) Erratum: [Phys. Lett. B **692**, 342 (2010)]; K. Nozari and A. Etemadi, Phys. Rev. D **85**, 104029 (2012).
- [22] A. F. Ali, S. Das and E. C. Vagenas, Phys. Rev. D **84**, 044013 (2011).
- [23] F. Marin *et al.*, Nature Phys. **9**, 71 (2013).
- [24] S. Das and E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008); P. Jizba, H. Kleinert and F. Scardigli, Phys. Rev. D **81**, 084030 (2010); K. Nozari and P. Pedram, Europhys. Lett. **92**, 50013 (2010); P. Pedram, K. Nozari and S. H. Taheri, JHEP **1103**, 093 (2011); P. Wang, H. Yang and X. Zhang, Phys. Lett. B **718**, 265 (2012); I. Pikovski, M. R. Vanner, M. Aspelmeyer, M. S. Kim and C. Brukner, Nature Phys. **8**, 393 (2012); F. Marin *et al.*, Nature Phys. **9**, 71 (2013); S. Jalalzadeh, M. A. Gorji and K. Nozari, Gen. Rel. Grav. **46**, 1632 (2014); S. Ghosh, Class. Quant. Grav. **31**, 025025 (2014); K. Nozari, M. Khodadi and M. A. Gorji, Europhys. Lett. **112**, no. 6, 60003 (2015); F. Scardigli and R. Casadio, Eur. Phys. J. C **75**, no. 9, 425 (2015); M. Khodadi, Astrophys. Space Sci. **358**, no. 2, 45 (2015).
- [25] A. F. Ali and B. Majumder, Class. Quant. Grav. **31**, no. 21, 215007 (2014).
- [26] E. J. Copeland, A. R. Liddle and D. Wands, Phys. Rev. D **57**, 4686 (1998); P. G. Ferreira and M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997); X. m. Chen, Y. g. Gong and E. N. Saridakis, JCAP **0904**, 001 (2009); G. Leon and E. N. Saridakis, JCAP **1303**, 025 (2013).
- [27] Q. Huang, P. Wu and H. Yu, Phys. Rev. D **91**, no. 10, 103502 (2015).

